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Guaranteed RMS Estimates of Spectral Functions and Radiation Sources from Observations with Random Disturbances

Oleksandr Nakonechnyi[1]*, Petro Zinko[1], Taras Zinko[1], Mariia Losieva[1]

[1]Department of System Analysis and Decision Making Theory, Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Volodymyrska str. 64/13, Kyiv, Ukraine

*Correspondence:*oleksandrnakonechny@knu.ua*

Abstract

In this article, we investigate the problem of estimating the spectral characteristics of the intensity of moving objects under stochastic uncertainty based on the results of measuring the magnitudes of wave fields at separate points. We assume that the unknown spectral functions belong to known sets from the functional space, and certain restrictions are given for the unknown correlation matrices. For linear guaranteed estimates of the set of linear functionals from spectral functions, we prove that the guaranteed mean square estimates are expressed in terms of solutions of a certain system of linear algebraic equations.

KEY WORDS: *linear estimate of the vector, guaranteed estimate of the vector, vector estimation error, spectral function, moving radiation source, stochastic uncertainty, observation*.

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1. Introduction

Modern radar and radio navigation air traffic control systems of various purposes (ground, ship, aircraft, space-based) function under the influence of both unorganized (natural and artificial) and organized obstacles. Usually, the statistical and temporal characteristics of these obstacles are unknown and subject to change. Therefore, the task of detecting signals (or signal sources) increasingly has to be solved in conditions of statistical a priori uncertainty. This means that a number of parameters, including the functions of the distribution of interference and mixtures of signals with interference, are not known precisely and can change during the observation process.

This work is devoted to one of the directions of signal processing from moving sources in passive location under stochastic uncertainty. The relevance of such subject matter is due to the need to detect and evaluate the parameters of inconspicuous (both stationary and moving objects) against the background of disturbances, which are characterized by the absence of complete a priori data relative to observation errors and spectral characteristics. The development of mathematical methods in conditions of incomplete information is important both in the localization of objects and in their classification [1]. When solving such problems, it is necessary to take into account both the equation of signal propagation and a priori information about the parameters of objects. In this regard, it is relevant to develop algorithms for detecting signals (or signal sources) with a high degree of a priori uncertainty for application in noise locators, passive radar, meteorological locators, etc. In our article, we offer effective algorithms for obtaining guaranteed root mean square estimates of the spectral functions of signals and radiation sources based on observations with random disturbances.

2. The Mathematical Background

When the signal propagation equations are deterministic, the parameters can be estimated using random field analysis methods [2, 3], or estimation methods in inverse problems [4, 5, 6].

Spectral characteristics play a crucial role in object classification tasks. The evaluation is carried out using parametric and nonparametric methods [7, 8, 9, 10, 11, 12] that are based on the Fourier transform [13].

Under conditions of statistical uncertainty, there are different approaches to signal processing in order to evaluate their spectral functions. In linear processing problems with deterministic disturbance quantities, methods are used to minimize guaranteed estimation errors [14, 15].

In the case of signals with interference of a statistical nature with unknown distributions, methods are used to minimize guaranteed root mean square errors [16, 17, 18]. The Bayesian focusing method [19, 20, 21] is a method that can be used to solve parameter estimation problems in the passive location of radiation sources with random characteristics, but it requires knowledge of a priori distributions of random parameters.

In works [22, 23, 24, 25] machine learning, deep learning, sparse Bayesian learning, and artificial intelligence are employed for processing audio signals.

3. Main Results

Let the elements of the matrix $Y = \left(y_{sj}\right)_{s=\overline{1,N}}^{j=\overline{1,n}}$ be observed, where

$$y_{sj} = \sum_{k=1}^{m} E f_k\left(t_s - \tau_{sj}\left(k,\theta\right)\right)\frac{c}{\tau_{sj}\left(k,\theta\right)} + \xi_{sj},\ s=\overline{1,N},\ j=\overline{1,n}, \tag{1}$$

$$\tau_{sj}\left(k,\theta\right) = \left|r^{(j)} - r^{(k)}\left(t_s,\theta\right)\right|_1 / c,\ s=\overline{1,N},\ j=\overline{1,n}, k=\overline{1,m},$$

$$r_j = \left(x_1(j), x_2(j), x_3(j)\right),\ j=\overline{1,n};$$

$$r_k\left(t_s\right) = \left(x_1^{(k)}\left(t_s\right), x_2^{(k)}\left(t_s\right), x_3^{(k)}\left(t_s\right)\right),\ k=\overline{1,m},\ s=\overline{1,N},$$

$f_k(t), k=\overline{1,m}$ are the radiation intensity of the k-th moving source at the time t (scalar functions $f_k(t), t\in R^1$ are bounded);

$r^{(j)},\ j=\overline{1,n}$ are coordinates of the signal receivers;

$r^{(k)}\left(t_s,\theta\right), k=\overline{1,m}$ are the vector of coordinates of the k-th radiation source at time t_s, which depends on a vector random parameter $\theta\in R^{m_1}$ with a known distribution $P(\cdot)$ (and for each $k\in\overline{1,m}, s\in\overline{1,N}, j\in\overline{1,n}$ the inequality $E\tau_{zj}^{-1}\left(k,\theta\right) \triangleq \int_{R^{m_1}}\tau_{zj}^{-1}\left(k,\theta\right)P\left(d\theta\right) < \infty$ holds);

c is the signal propagation speed;

$t_s, s=\overline{1,N}, t_{s+1} > t_s$ are given sequence of discrete time instants;

$\xi_{sj}, s=\overline{1,N},\ j=\overline{1,n}$ are the observation errors, which are scalar random variables that have met the conditions $E\xi_{sj} = 0, s=\overline{1,N},\ j=\overline{1,n}$, and the values $r_{sjkl} \triangleq E\xi_{sj}\xi_{kl},\ s=\overline{1,N},\ j=\overline{1,n}, k=\overline{1,N}, l=\overline{1,n}$ are unknown.

Here E is the sign of the mathematical expectation, and the length of the vector $r = \left(x_1, x_2, x_3\right)$ is determined by the formula $|r|_1 \triangleq \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Let's assume that scalar functions $f_k(t), t\in R^1$ allow a representation

$$f_k(t) = \int_{R^1}\exp(-it\omega)\,\tilde{f}_k(\omega)\,d\omega, \quad t\in R^1,$$

where $\tilde{f}_k(\omega), k=\overline{1,m}$ are integrated with the square unknown functions. This means that inequalities $\int_{R^1}\left|\tilde{f}_k(\omega)\right|^2 d\omega < \infty$, hold for all $k=\overline{1,m}$. Here the sign $|\cdot|$ is modulus of a complex number.

Our objective is to obtain estimates of certain functionals from unknown functions $\tilde{f}_k(\omega), k=\overline{1,m}$ through observations (1).

Let the following form be used for recording observations (1)

$$Y = \int_{R^1} F(\omega)\,\tilde{f}(\omega)\,d\omega + \xi, \tag{2}$$

where $F(\omega)\,f(\omega) \triangleq \sum_{k=1}^{m} F_k(\omega)\tilde{f}_k(\omega),\ F_k(\omega), k=\overline{1,m}$ are sequence of matrices of the form

$$F_k(\omega) \square \left(E\exp\left(-i\omega\left(t_s - \tau_{sj}(k,\theta)\right)\right)\frac{c}{\tau_{sj}(k,\theta)}\right)_{s=\overline{1,N}}^{j=\overline{1,n}};$$

ξ is a random matrix with the elements $\xi_{sj}, s=\overline{1,N}, j=\overline{1,n}$.

We assume that the unknown vector functions $\tilde{f}(\omega) \square \left(\tilde{f}_1(\omega), \tilde{f}_2(\omega),..., \tilde{f}_m(\omega)\right)^T$ belong to the set

$$G_1 \square \left\{\tilde{f} : \int_{R^1}\left(Q_1(\omega)\left(\tilde{f}(\omega) - f_0(\omega)\right), \tilde{f}(\omega) - f_0(\omega)\right)d\omega \le 1\right\}, \quad (3)$$

where the matrix $Q_1(\omega), \omega \in R^1$ is defined by the rule $\left(Q_1(\omega)\tilde{f}(\omega), \tilde{f}(\omega)\right) \square \sum_{k=1}^m q_k^2(\omega)\left|\tilde{f}_k(\omega)\right|^2$, scalar functions $q_k(\cdot), k=\overline{1,m}$ satisfy the inequalities $\int_{R^1} q_k^{-2}(\omega)d\omega < \infty, k=\overline{1,m}$; $f_0(\omega), \omega \in R^1$ is known vector function such that $\int_{R^1}|f_0(\omega)|^2 d\omega < \infty$.

Let us denote by R the correlation operator of the matrix ξ. Let us introduce the scalar product in the space of linear operators that act from a space of $N \times n$ matrices into the same space of matrices. If R_1 and R_2 are two such linear operators, that are defined by the elements $r_{sjkl}^{(1)}, s=\overline{1,N}, j=\overline{1,n}, k=\overline{1,N}, l=\overline{1,n}$ and $r_{sjkl}^{(2)}, s=\overline{1,N}, j=\overline{1,n}, k=\overline{1,N}, l=\overline{1,n}$ respectively, then the scalar product is defined by the rule $\langle R_1, R_2\rangle_2 \square \sum_{s,j,k,l} r_{sjkl}^{(1)} r_{sjkl}^{(2)}$.

We define the scalar product in the space of $N \times n$ matrices by the rule $\langle A, B\rangle_1 \square \sum_{s,j} a_{sj}\bar{b}_{sj}$, where $A \square (a_{sj})_{s=\overline{1,N}}^{j=\overline{1,n}}$, $B \square (b_{sj})_{s=\overline{1,N}}^{j=\overline{1,n}}$, $\bar{b}_{sj}$ complex number adjoint to b_{sj}.

Let the components of complex-valued vector functions $l^{(q)}(\omega) \square \left(l_1^{(q)}(\omega), l_2^{(q)}(\omega),..., l_m^{(q)}(\omega)\right), \omega \in R^1, q=\overline{1,p}$ satisfy the inequalities

$$\int_{R^1}\left|l_k^{(q)}(\omega)\right|^2 d\omega < \infty,\ k=\overline{1,m}, q=\overline{1,p}.$$

We also introduce a linear operator L according to the rule

$$L\tilde{f} \square \left(\left(l^{(1)}, \tilde{f}\right), \left(l^{(2)}, \tilde{f}\right),..., \left(l^{(p)}, \tilde{f}\right)\right)^T,$$

where $\left(l^{(q)}, \tilde{f}\right) \square \int_{R^1}\left(l^{(q)}(\omega), \tilde{f}(\omega)\right)d\omega,\ q=\overline{1,p};\quad \left(l^{(q)}(\omega), \tilde{f}(\omega)\right) \square \sum_{k=1}^m l_k^{(q)}(\omega)\,\bar{\tilde{f}}_k(\omega), \omega \in R^1,\ q=\overline{1,p}$.

Definition 1. Vector $L\tilde{f}$ is called *a linear estimate of vector* $L\tilde{f}$ by observations (2) if it has the form

$$L\tilde{f} \square \left(\langle U_1, Y\rangle_1 + c_1, \langle U_2, Y\rangle_1 + c_2,..., \langle U_p, Y\rangle_1 + c_p\right)^T,$$

where $U_1, U_2,..., U_p$ are some $N \times n$ matrices complex-valued elements, $c_1, c_2,..., c_p$ are some complex numbers.

Let the correlation operator R belong to a certain set G_2.

Definition 2. The linear estimate $L\tilde{f}$ is called *a linear guaranteed estimate of the vector* $L\tilde{f}$ if it has the form

$$L\tilde{f} \square \left(\langle \hat{U}_1, Y\rangle_1 + c_1, \langle \hat{U}_2, Y\rangle_1 + c_2,..., \langle \hat{U}_p, Y\rangle_1 + c_p\right),$$

and determined from the condition

$$\min_{c,U_1,...,U_p}\max_{G_1,G_2}\sum_{q=1}^p E\left|\left(l^{(q)}, \tilde{f}\right) - \langle U_q, Y\rangle_1 - c_q\right|^2 =$$

$$\max_{G_1,G_2}\sum_{q=1}^p E\left|\left(l^{(q)}, \tilde{f}\right) - \langle \hat{U}_q, Y\rangle_1 - \hat{c}_q\right|^2,$$

where $c=\left(c_1,c_2,...,c_p\right)$.

Theorem 1. *Let the set* G_2 *have the form*

$$G_2 := \left\{R : E\left\langle Q_2\xi,\xi\right\rangle_1 \le 1\right\}, \tag{4}$$

where Q_2 *is known positive-definite operator. Then the equality holds*

$$\begin{aligned}&\min_c \max_{G_1,G_2} \sum_{q=1}^{p} E\left|\left(l^{(q)},\tilde f\right)-\left\langle U_q,Y\right\rangle_1-c_q\right|^2 = \\ &\lambda_{\max}\left(A\left(U_1,U_2,...,U_p\right)\right)+\lambda_{\max}\left(B\left(U_1,U_2,...,U_p\right)\right),\end{aligned} \tag{5}$$

where matrices $A\left(U_1,U_2,...,U_p\right)=\left(a_{qj}\right)_{q=\overline{1,p}}^{j=\overline{1,p}}$, $B\left(U_1,U_2,...,U_p\right)=\left(b_{qj}\right)_{q=\overline{1,p}}^{j=\overline{1,p}}$ *are defined by the rule*

$$\begin{aligned}a_{qj}\left(U_1,U_2,...,U_p\right)&=\int_{R^1}\left(Q_1^{-1}(\omega)\varphi_q(\omega),\varphi_j(\omega)\right)d\omega,\\ \varphi_q(\omega)&=l^{(q)}(\omega)-F^*(\omega)U_q,\\ b_{qj}\left(U_1,U_2,...,U_p\right)&=\left\langle Q_2U_q,U_j\right\rangle_1,\ q,j=\overline{1,p},\end{aligned}$$

and $F^*(\omega)$ *is the conjugate operator of* $F(\omega)$.

Proof. Let us denote by $\delta^2\left(U_1,U_2,...,U_p\right)$ the following expression $\delta^2\left(U_1,U_2,...,U_p\right) := \min\limits_c \max\limits_{G_1,G_2} \sum_{q=1}^{p} E\left|\left(l^{(q)},\tilde f\right)-\left\langle \hat U_q,Y\right\rangle_1-c_q\right|^2$. It is clear that the equality holds

$$\begin{aligned}&\delta^2\left(U_1,U_2,...,U_p\right) := \\ &\min_c \max_{\sum_{q=1}^{p}\left(\alpha_q\right)^2=1} \max_{G_1}\left(\operatorname{Re}\int_{R^1}\left(\sum_{q=1}^{p}\alpha_q\varphi_q(\omega),\tilde f(\omega)-f_0(\omega)\right)d\omega+\right.\\ &\left.\operatorname{Re}\left(\sum_{q=1}^{p}\int_{R^1}\left(\alpha_q\varphi_q(\omega),f_0(\omega)\right)d\omega-\sum_{q=1}^{p}\alpha_q c_q\right)\right)^2+\\ &\max_{G_2}\sum_{q=1}^{p}E\left|\left\langle U_q,\xi\right\rangle_1\right|^2,\end{aligned}$$

and inequalities

$$\begin{aligned}&\left(\operatorname{Re}\int_{R^1}\left(\sum_{q=1}^{p}\alpha_q\varphi_q(\omega),\tilde f(\omega)-f_0(\omega)\right)d\omega\right)^2\le\\ &\le\int_{R^1}\sum_{q=1}^{p}\left(\alpha_q\varphi_q(\omega),Q_1^{-1}(\omega)\sum_{q=1}^{p}\alpha_q\varphi_q(\omega)\right)d\omega\times\\ &\int_{R^1}\left(Q_1(\omega)\left(\tilde f(\omega)-f_0(\omega)\right),\tilde f(\omega)-f_0(\omega)\right)d\omega\le\\ &\le\int_{R^1}\left(\sum_{q=1}^{p}\alpha_q\varphi_q(\omega),Q_1^{-1}(\omega)\sum_{q=1}^{p}\alpha_q\varphi_q(\omega)\right)d\omega,\end{aligned}$$

are converted into equalities when the next condition is met

$$\begin{aligned}&\tilde f(\omega)=f_0(\omega)+\\ &Q_1^{-1}(\omega)\varphi(\omega)\left(\int_{R^1}\left(Q_1^{-1}(\omega)\varphi(\omega),\varphi(\omega)\right)d\omega\right)^{-1/2},\end{aligned}$$

where $\varphi(\omega) := \sum_{q=1}^{p}\alpha_q\varphi_q(\omega)$. Then we get the following representation for the functional $\delta^2\left(U_1,U_2,...,U_p\right)$:

$$\begin{aligned}&\delta^2\left(U_1,U_2,...,U_p\right)=\\ &\lambda_{\max}\left(A\left(U_1,U_2,...,U_p\right)\right)+\max_{G_2}\sum_{q=1}^{p}E\left|\left\langle U_q,\xi\right\rangle_1\right|^2.\end{aligned}$$

Note that the following relations are valid

$$\max_{G_2}\sum_{q=1}^{p}E\left|\left\langle U_q,\xi\right\rangle_1\right|^2=\max_{\bar G_2}\sum_{q=1}^{p}\left|\left\langle U_q,W\right\rangle_1\right|^2=$$
$$\max_{\bar G_2}\max_{\sum_{q=1}^{p}|\beta_q|^2=1}\left(\operatorname{Re}\left\langle\sum_{q=1}^{p}\beta_qU_q,W\right\rangle_1\right)^2,$$

where $\bar G_2 \square \left\{W:\langle Q_2W,W\rangle_1\le 1\right\}$. Hence, we get the inequality

$$\operatorname{Re}\left(\left\langle\sum_{q=1}^{p}\beta_qU_q,W\right\rangle_1\right)^2\le\left\langle Q_2^{-1}\sum_{q=1}^{p}\beta_qU_q,\sum_{q=1}^{p}\beta_qU_q\right\rangle_1. \tag{6}$$

In relation (6), equality is achieved with the following value of the matrix $W=\hat W$:

$$\hat W=Q_2^{-1}\left(\sum_{q=1}^{p}\beta_qU_q\right)\left\langle Q_2^{-1}\left(\sum_{q=1}^{p}\beta_qU_q\right),\left(\sum_{q=1}^{p}\beta_qU_q\right)\right\rangle_1^{-1/2}\eta,$$

where η is the random variable such that $E\eta=0,\ E\eta^2=1$. Now we can write the equality

$$\max_{G_2}\sum_{q=1}^{p}E\left|\left\langle U_q,\xi\right\rangle_1\right|^2=\lambda_{\max}\left(B\left(U_1,U_2,...,U_p\right)\right)$$

and conclude that formula (5) is valid. □

Corollary 1. Let the conditions of Theorem 1 be satisfied and $\left(\hat U_1,\hat U_2,...,\hat U_p\right)\in Arg\min_{U_1,...,U_p}\delta^2\left(U_1,U_2,...,U_p\right)$; $\hat\alpha=\left(\hat\alpha_1,\hat\alpha_2,...,\hat\alpha_p\right)$ and $\hat\beta=\left(\hat\beta_1,\hat\beta_2,...,\hat\beta_p\right)$ are the eigenvectors corresponding to the maximal eigenvalues of the matrices $A\left(\hat U_1,\hat U_2,...,\hat U_p\right)$ and $B\left(\hat U_1,\hat U_2,...,\hat U_p\right)$ respectively. Then the equality holds:

$$\max_{G_1,G_2}\delta_1^2\left(\tilde f,\varsigma\right)=\delta_1^2\left(\tilde f_g,\varsigma_g\right),$$

where $\delta_1^2\left(\tilde f,\varsigma\right)=\sum_{q=1}^{p}\left|\int_{R^1}\left(\hat\varphi_q(\omega),\tilde f(\omega)\right)d\omega\right|^2+\sum_{q=1}^{p}E\left|\left\langle\hat U_q,\varsigma\right\rangle_1\right|^2$,

$$\tilde f_g(\omega)=f_0(\omega)+Q_1^{-1}(\omega)\hat\varphi(\omega)\left(\int_{R^1}\left(Q_1^{-1}(\omega)\hat\varphi(\omega),\hat\varphi(\omega)\right)d\omega\right)^{-1/2},$$

$$\varsigma_g=\eta Q_2^{-1}\hat U\left(\left\langle Q_2^{-1}\hat U,\hat U\right\rangle_1\right)^{-1/2},\ \hat\varphi(\omega)=\sum_{q=1}^{p}\hat\alpha_q\hat\varphi_q(\omega),$$

$$\hat U=\sum_{q=1}^{p}\hat\beta_q\hat U_q,\ \hat\varphi_q(\omega)=l^{(q)}(\omega)-F^*(\omega)\hat U_q,$$

η is the random variable such that $E\eta=0,\ E\eta^2=1$.

Corollary 2. Let the conditions of Theorem 1 be satisfied and parameter $p=1$. Then, for the squared error of the functional estimate, the following equalities:

$$\sigma_g^2=\min_{c_1,U_1}\max_{G_1,G_2}E\left|\int_{R^1}\left(l^{(1)}(\omega),\tilde f(\omega)\right)d\omega-\langle U_1,Y\rangle_1-c_1\right|^2=$$
$$\max_{G_1,G_2}E\left|\int_{R^1}\left(l^{(1)}(\omega),\tilde f(\omega)\right)d\omega-\left\langle\hat U_1,Y\right\rangle_1-\hat c_1\right|^2=$$
$$\left|\int_{R^1}\left(\hat\varphi_1(\omega),f_g(\omega)\right)d\omega\right|^2+E\left|\left\langle\hat U_1,\varsigma_g\right\rangle_1\right|^2,$$

are holds, where $\hat U_1$ is solution of the matrix equation

$$Q_2^{-1}\hat U_1+\int_{R^1}F(\omega)Q_1^{-1}(\omega)F^*(\omega)d\omega\hat U_1=$$
$$\int_{R^1}F(\omega)Q_1^{-1}(\omega)l^{(1)}(\omega)d\omega \tag{7}$$

and at the same time $\hat{c}_1=\int_{R^1}\left(\hat{\varphi}_1(\omega), f_0(\omega)\right)d\omega$, $\varsigma_g(\omega)=Q_2^{-1}\hat{U}_1\left(\left\langle Q_2^{-1}\hat{U}_1,\hat{U}_1\right\rangle\right)^{-1/2}\eta$, $f_g(\omega)=f_0(\omega)+Q_1^{-1}(\omega)\hat{\varphi}_1(\omega)\times$ $\times\left(\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),\hat{\varphi}_1(\omega)\right)d\omega\right)^{-1/2}$, where η is the random variable such that $E\eta=0$, $E\eta^2=1$.

Proof. Let us denote by $\Phi_1(U_1)$ the following functional

$$\Phi_1(U_1)=\min_{c_1}\max_{G_1,G_2} E\left|\int_{R^1}\left(l^{(1)}(\omega), \tilde{f}(\omega)\right)d\omega-\langle U_1,Y\rangle_1-c_1\right|^2.$$

Then, as follows from Theorem 1, the functional $\Phi_1(U_1)$ can be represented in the form

$$\Phi_1(U_1)=\int_{R^1}\left(Q_1^{-1}(\omega)\varphi_1(\omega),\varphi_1(\omega)\right)d\omega+\left\langle Q_2^{-1}U_1,U_1\right\rangle_1.$$

The functional $\Phi_1(U_1)$ has a unique minimum $\hat{U}_1$, which is found from the condition $\frac{d}{d\tau}\Phi_1\left(\hat{U}_1+\tau V\right)\big|_{\tau=0}\equiv 0$ for an arbitrary $N\times n$ matrix V. Thus, to find the matrix $\hat{U}_1$ we obtain equation (7). The expressions for calculating vectors $f_g(\omega)$ and ς_g are a special case of the corresponding expressions for vectors $f_g(\omega)$ and ς_g from Corollary 1 to Theorem 1. □

Theorem 2. *When the parameter $p=1$, the equality holds*

$$\sigma_g^2=\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),l^{(1)}(\omega)\right)d\omega.$$

Proof. For an arbitrary $N\times n$ matrix V the equalities hold

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\Phi_1\left(\hat{U}_1+\tau V\right)\big|_{\tau=0}=\\&-\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),F^*(\omega)V\right)d\omega+\mathrm{Re}\left\langle Q_2^{-1}\hat{U}_1,V\right\rangle_1\equiv 0.\end{aligned}\qquad(8)$$

Let us put in formula (8) $V=\hat{U}_1$. Then from (8) we obtain the equality

$$\left\langle Q_2^{-1}\hat{U}_1,\hat{U}_1\right\rangle_1=\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),F^*(\omega)\hat{U}_1\right)d\omega.$$

Thus, for the square of the error the following expressions are valid

$$\begin{aligned}&\sigma_g^2=\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),\hat{\varphi}_1(\omega)\right)d\omega+\left\langle Q_2\hat{U}_1,\hat{U}_1\right\rangle_1=\\&\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),l^{(1)}(\omega)-F^*(\omega)\hat{U}_1\right)d\omega+\\&+\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),F^*(\omega)\hat{U}_1\right)d\omega=\\&\mathrm{Re}\int_{R^1}\left(Q_1^{-1}(\omega)\hat{\varphi}_1(\omega),l^{(1)}(\omega)\right)d\omega.\end{aligned}$$

Theorem 2 is proved. □

Theorem 3. *When the parameter $p=1$, the equality*

$$\left\langle \hat{U}_1,Y\right\rangle_1+\hat{c}_1=\int_{R^1}\left(l^{(1)}(\omega), \tilde{f}(\omega)\right)d\omega,$$

holds, where $\tilde{f}(\omega)=\Gamma(\omega)X+f_0(\omega)$, $\Gamma(\omega)=Q_1^{-1}(\omega)F^*(\omega)$ and X is a solution to the matrix equation

$$Q_2^{-1}X+\int_{R^1}F(\omega)Q_1^{-1}(\omega)F^*(\omega)d\omega X=Y-\int_{R^1}F(\omega)f_0(\omega)d\omega.\qquad(9)$$

Proof. The following equalities

$$\left\langle \hat{U}_1, Y \right\rangle_1 + \hat{c}_1 = \left\langle \hat{U}_1, Y \right\rangle_1 + \int_{R^1} \left(l^{(1)}(\omega) - F^*(\omega)\hat{U}_1, f_0(\omega) \right) d\omega =$$
$$\left\langle \hat{U}_1, Y - \int_{R^1} F(\omega) f_0(\omega) d\omega \right\rangle_1 + \int_{R^1} \left(l^{(1)}(\omega), f_0(\omega) \right) d\omega \qquad (10)$$

are holds. From equations (7), (9) we obtain the following equalities

$$\left\langle \hat{U}_1, Y - \int_{R^1} F(\omega) f_0(\omega) d\omega \right\rangle_1 =$$
$$\left\langle \int_{R^1} F(\omega) Q_1^{-1}(\omega) l^{(1)}(\omega) d\omega, X \right\rangle_1 =$$
$$\int_{R^1} \left(Q_1^{-1}(\omega) F^*(\omega) X, l^{(1)}(\omega) \right) d\omega. \qquad (11)$$

From equalities (10), (11) follows $\left\langle \hat{U}_1, Y \right\rangle_1 + \hat{c}_1 = \int_{R^1} \left(l^{(1)}(\omega), Q_1^{-1}(\omega) F^*(\omega) X + f_0(\omega) \right) d\omega = \int_{R^1} \left(l^{(1)}(\omega), \tilde{f}(\omega) \right) d\omega$, which completes the proof of Theorem 3. □

Let $\psi_q(\omega), \omega \in R^1, q = 1, 2, ...$ be an ortho-normalized basis in the space of square-integrable complex-valued vector functions. Therefore, for an arbitrary function $\tilde{f}(\omega), \omega \in R^1$ from this space, the inequality $\int_{R^1} \left(\tilde{f}(\omega), \tilde{f}(\omega) \right) d\omega < \infty$ holds, and representation

$$\tilde{f}(\omega) = \sum_{q=1}^{\infty} \left(\tilde{f}(\cdot), \psi_q(\cdot) \right) \psi_q(\omega), \ \omega \in R^1,$$

is valid, where $\left(\tilde{f}(\cdot), \psi_q(\cdot) \right) = \int_{R^1} \left(\tilde{f}(\omega), \psi_q(\omega) \right) d\omega$.

Let the expression $\hat{\tilde{f}}^{(p)}(\omega) ≔ \sum_{q=1}^{p} \left(\tilde{f}(\cdot), \psi_q(\cdot) \right)_g \psi_q(\omega), \omega \in R^1$ is used as the linear estimate of the vector function $\tilde{f}^{(p)}(\omega) ≔ \sum_{q=1}^{p} \left(\tilde{f}(\cdot), \psi_q(\cdot) \right) \psi_q(\omega), \omega \in R^1$ from observations (2), where $\left(\tilde{f}(\cdot), \psi_q(\cdot) \right)_g, q = \overline{1, p}$ are linear guaranteed mean square estimates of scalar products $\left(\tilde{f}(\cdot), \psi_q(\cdot) \right), q = \overline{1, p}$, that have the form $\left(\tilde{f}(\cdot), \psi_q(\cdot) \right)_g ≔ \hat{U}(q) Y + \hat{c}_1(\psi_q) = \int_{R^1} \left(\psi_q(\omega), \hat{\tilde{f}}^{(p)}(\omega) \right) d\omega$. Here $\hat{U}(q)$ is the solution of equation (7) when $l^{(1)}(\omega) = \psi_q(\omega)$, $\hat{c}_1(\psi_q) = \int_{R^1} \left(\psi_q(\omega) - F^*(\omega)\hat{U}(q), f_0(\omega) \right) d\omega$.

Theorem 4. *The equality*

$$\max_{G_1, G_2} \int_{R^1} E\left(\tilde{f}^{(p)}(\omega) - \hat{\tilde{f}}^{(p)}(\omega), \tilde{f}^{(p)}(\omega) - \hat{\tilde{f}}^{(p)}(\omega) \right) d\omega =$$
$$\lambda_{\max}\left(A_1\left(\hat{U}(1), \hat{U}(2), ..., \hat{U}(p) \right) \right) +$$
$$+ \lambda_{\max}\left(B_1\left(\hat{U}(1), \hat{U}(2), ..., \hat{U}(p) \right) \right)$$

holds, where $A_1(\cdot)$ and $B_1(\cdot)$ are $p \times p$ matrices with elements respectively,
$a^{(1)}_{q_1, q_2} = \int_{R^1} \left(\psi_{q_1}(\omega) - F^*(\omega)\hat{U}(q_1), Q_1^{-1}\left(\psi_{q_2}(\omega) - F^*(\omega)\hat{U}(q_2) \right) \right) d\omega$, $b^{(1)}_{q_1, q_2} = \left\langle Q_2^{-1} \hat{U}(q_1), \hat{U}(q_2) \right\rangle_1$, $q_1, q_2 = \overline{1, p}$.

The validity of this theorem follows from Theorem 1 and Corollaries 1, 2.

4. Numerical Experiment

Computation of the spectral estimate of a moving signal-emitting source for $m = 1$. We specify the signal in the form

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(t - a)^2 \right), \ t \in R^1, \ \sigma > 0, \ a \in R^1.$$

The spectral function of this signal is:

$$\tilde{f}(\omega)=\exp\left(i\omega a-\frac{1}{2}\sigma^2\omega^2\right),\ \omega\in R^1.$$

We find the spectral function magnitude:

$$\left|\tilde{f}(\omega)\right|=\frac{1}{2\pi}\exp\left(-\frac{1}{2}\sigma^2\omega^2\right),\ \omega\in R^1.$$

We specify the sequence of discrete moments of time $t_k=a-3\sigma+\Delta t\cdot(k-1),\ k=\overline{1,N}$. For the selected values of the natural number N and the parameter $\sigma>0$, the discretization step Δt is calculated by the formula

$$\Delta t=6\sigma/(N-1).$$

Denote by $r^{(0)}=\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)$ the coordinates of the signal receiver and by $r^{(1)}(t_k)=\left(x_1^{(1)}(t_k),x_2^{(1)}(t_k),x_3^{(1)}(t_k)\right),k=\overline{1,N}$ the coordinates of the moving signal source at time instants t_k , $k=\overline{1,N}$.

Let's define the values $\tau_k=\left|r^{(0)}-r^{(1)}(t_k)\right|/c=\sqrt{\left(x_1^{(0)}-x_1^{(1)}(t_k)\right)^2+\left(x_2^{(0)}-x_2^{(1)}(t_k)\right)^2+\left(x_3^{(0)}-x_3^{(1)}(t_k)\right)^2}/c$, $k=\overline{1,N}$, where $x_3^{(1)}(t_k)=const=\chi,k=\overline{1,N}$, $x_s^{(1)}(t_k)=\chi-v_s\cdot(t_k+3\sigma),\chi\in R^1,v_s>0,s=\overline{1,2},k=\overline{1,N}$, c is the speed of sound.

Let's introduce variables $\overline{t}_k=t_k-\tau_k$, $k=\overline{1,N}$.

To construct an estimate $\left|\hat{\tilde{f}}(\omega)\right|$ of the spectral function magnitude $\tilde{f}(\omega),\ \omega\in R^1$, we use formula (9) from Theorem 3 when $f_0(\omega)\equiv 0,\omega\in R^1$:

$$Q_2^{-1}X+\int_{R^1}F(\omega)Q_1^{-1}(\omega)F^*(\omega)d\omega\cdot X=Y, \tag{12}$$

where $X=(x_1,x_2,...,x_N)^T$ is the unknown column vector, $Y=(y_1,y_2,...,y_N)^T$ is the known column vector of signal measurements $y_k=\dfrac{1}{\sigma\sqrt{2\pi}\exp\left((\overline{t}_k-a)^2/(2\sigma^2)\right)},k=\overline{1,N}$,

$$Q_1^{-1}(\omega)=\frac{1}{1+\omega^2},\quad Q_2^{-1}=\delta^2>0,$$

$$F(\omega)=\left(\exp(-i\omega\overline{t}_k)\right)_{k=\overline{1,N}},\quad F^*(\omega)=\left(\exp(i\omega\overline{t}_k)\right)_{k=\overline{1,N}}.$$

Thus, from (12), a system of linear algebraic equations for determining the vector X is obtained

$$\delta^2IX+BX=Y, \tag{13}$$

where I is unit $N\times N$ matrix, $B=(b_{kj})_{k=\overline{1,N}}^{j=\overline{1,N}}$, $b_{kj}=\dfrac{\pi}{\exp\left(\left|\overline{t}_k-\overline{t}_j\right|\right)}$.

Using Theorem 3, we find the expression for the estimation of the spectral function:

$$\begin{aligned}\hat{\tilde{f}}(\omega)&=\frac{1}{1+\omega^2}\sum_{j=1}^N x_j\exp\left(i\omega\overline{t}_j\right)=\\&=\frac{1}{1+\omega^2}\left(\sum_{j=1}^N x_j\cos\left(\omega\overline{t}_j\right)+i\sum_{j=1}^N x_j\sin\left(\omega\overline{t}_j\right)\right),\omega\in R^1.\end{aligned} \tag{14}$$

The spectral function magnitude estimation (14) is given by

$$\left|\hat{\tilde{f}}(\omega)\right|=\frac{1}{1+\omega^2}\left(\sum_{k=1}^N\sum_{j=1}^N x_kx_j\cos\left((\overline{t}_k-\overline{t}_j)\omega\right)\right)^{1/2},\omega\in R^1.$$

For calculations, we set the following parameter values:

$$c=332,\ x_s^{(0)}=0,\ \ s=\overline{1,3},$$

$$\chi=500,\ \ v_1=v_2=50,$$

$$\sigma=2,\ \ a=0,\ \ \delta^2\in\{0;\ 0.1;\ 0.5;\ 1\}.$$

Figure 1 shows the graphs of the functions $\left|\tilde{f}(\omega)\right|$ and $\left|\hat{\tilde{f}}(\omega)\right|$ on the segment $\omega \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ for different values of the parameter δ^2.

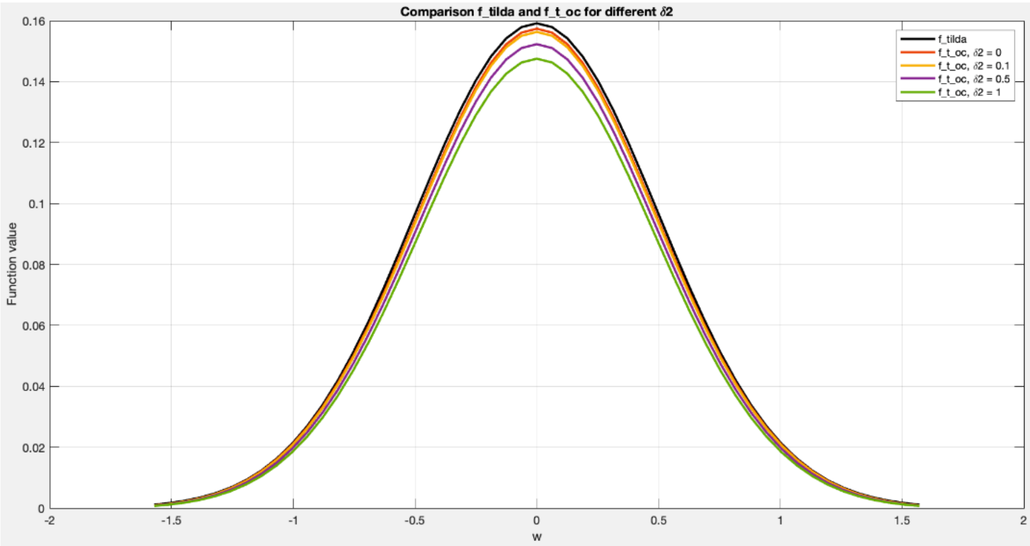

Fig.1. Function graphs: $\left|\tilde{f}(\omega)\right|$ is black, $\left|\hat{\tilde{f}}(\omega)\right|$ at $\delta^2 = 0$ is orange, at $\delta^2 = 0.1$ is yellow, at $\delta^2 = 0.5$ is violet, at $\delta^2 = 1$ is green.

Finally, we provide the table of values for the relative error of the spectral function magnitude estimation, calculated using the formula

$$\varphi\left(\delta^2, \omega_s\right) = \frac{\left|\tilde{f}(\omega_s)\right| - \left|\hat{\tilde{f}}(\omega_s)\right|}{\left|\tilde{f}(\omega_s)\right|},$$

$$\omega_s = (s-1)\frac{\pi}{8},\ s = \overline{1,5}.$$

Table 1.

Relative error values $\varphi\left(\delta^2, \omega_s\right)$.

$\omega_s \setminus \delta^2$	**0**	**0.1**	**0.5**	**1**
0	0.011	0.018	0.043	0.073
$\pi/8$	0.021	0.028	0.049	0.091
$\pi/4$	0.022	0.033	0.072	0.117
$3\pi/8$	0.020	0.035	0.092	0.154
$\pi/2$	0.198	0.217	0.286	0.355

Based on the Table 1, it can be concluded that the relative estimation error of the spectral function magnitude deteriorates with increasing signal measurement errors and higher values of $|\omega|$.

5. Conclusions

Effective algorithms are proposed for finding guaranteed root mean square estimates of the spectral functions of moving signal sources and radiation sources based on matrix observations with random disturbances. For linear guaranteed estimates of certain linear functionals from spectral functions, it is proved that guaranteed root mean square estimates are expressed in terms of solutions of defined systems of linear algebraic equations. Theoretical statements are confirmed by a test case.

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