Proceedings of 5th International Conference CNDGS'2026
ISSN 2669-2023 (print) / ISSN 2538-8959 (on-line) / **https://doi.org/10.47459/cndcgs.2026.91**

Forecasting Defense Spending of Conflicting Parties under Uncertainty

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Abstract

This paper investigates the problem of constructing predicted estimates for dynamic systems described by difference equations under uncertainty. In particular, the Richardson Arms Model is considered for two competing countries, which can be used to describe the dynamics of their armaments. The main attention is paid to the case when the armament level of one of the countries under consideration is partially unknown. The function describing this level belongs to given compact and convex sets, while the values of the system variables are known only at certain discrete moments of time. An approach is proposed to find the optimal strategy of the first country, which guarantees the achievement of a given result with the maximum uncertainty value with respect to the second country. Analytical relations for determining the maximum values of uncertain quantities are obtained and substantiated. The developed approach is based on the use of properties of convex sets, vector representations, and methods of analysis of dynamic systems. A number of statements and their consequences are proved, which ensure the efficient calculation of estimates in the presence of additional restrictions on the system parameters. The proposed approach can be applied to modeling and analyzing processes of population dynamics, information dissemination, and other socio-economic systems in conditions of incomplete information regarding the state of the system.

KEY WORDS: uncertainty; Richardson Arms Model; difference equations; convex sets; optimal strategy.

Citation: Nakonechnyi, O.; Kapustian, O.; Shakotko, T. Richardson Arms Model under with Partially Unknown Parameters. In Proceedings of the Challenges to National Defence in Contemporary Geopolitical Situation, Brno, Czech Republic, 7-10 September 2026. ISSN 2538-8959, https://doi.org/10.47459/cndcgs.2026.91

1. Introduction

The arms race model was developed by the British mathematician Lewis F. Richardson to study the dynamics of conflict between two countries in the defence sphere [1, 2]. Depending on the relationships between the parameters of the differential equations governing this dynamic, qualitatively different scenarios may arise in the interactions between the two countries — ranging from stable disarmament to an unbounded arms race [3, 4]. This model is fundamental for the mathematical modeling of international conflicts and for forecasting military confrontations [5, 6], and also finds application in the description of population dynamics, information dissemination, and other socio-economic processes [7, 8]. Questions of military spending convergence across countries and their impact on international security are investigated in [9, 10], while the stability and structure of alliances between major powers and weaker states is examined in [11]. The classical Richardson model is formulated as a system of differential equations; however, for practical problems of analysis and prediction, the discrete analogue — a system of difference equations — is more natural and computationally tractable. In real conflict situations, information concerning the state of one of the parties, in particular the level of its armaments, is only partially available, which gives rise to an optimal control problem under uncertainty. In the solution of such problems, criteria that account for fuzziness in the set of possible states of nature are widely applied [12], as well as game-theoretic equilibrium methods for fuzzy strategies [13]. Data-driven methods for modeling militarized interstate disputes and predicting conflicts are employed in [8], and threats of destabilization arising from geopolitical tensions are examined in [14].

In the present paper, a discrete-time version of Richardson's model is considered, in which the armament level of the second country is a partially unknown function belonging to given compact convex sets, while the values of the system variables are known only at certain discrete moments of time. The aim of this study is to find the optimal strategy of the first country that guarantees the achievement of a prescribed outcome under the maximum degree of uncertainty regarding the second country. The methodological basis of the approach comprises properties of convex sets, vector analysis methods, the theory of antagonistic games, and stability estimation methods that have been tested in the analysis of stress and psychological states under uncertainty [7, 15, 16] and in the study of neurodynamic systems with aftereffect [17, 18, 19].

2. The Mathematical Background

Studies of Richardson's mathematical arms race models were carried out in works [3, 4, 5]. Depending on the relationships between the parameters, the asymptotic behavior and qualitative properties of the trajectories of the equations were investigated — in particular, conditions for equilibrium and stability of the systems, as well as the long-time behavior of solutions [3]. Extensions to the case of more than two actors and time-integrated variants of the model were addressed in [4, 6]. The classical Richardson models were further extended by introducing time delays, which allow one to account for the inherent inertia of defence decision-making processes. Research in this direction was conducted in the works of A.V. Shatyrko and D.Ya. Khusainov [20, 21], where stability conditions and convergence properties of delayed systems were established. The theoretical methods applied to such systems are closely connected with approaches to the construction of Lyapunov-Krasovsky functions [18] and to the study of convergence of learning processes in neurodynamic models with aftereffect [17, 19].

In studying arms race models with uncertain parameters, either methods of Game Theory or methods of Stochastic Analysis were applied [22]. In the game-theoretic setting, each country is treated as a player with its own strategy set, and the Nash equilibrium serves as the solution concept [13]. However, in conditions where agreements between the parties are not observed, Nash strategies become ineffective, and cautious (guaranteed) strategies are used instead. To account for fuzziness in the set of possible states of nature, criteria based on a generalization of Germeyer's principle are employed [12]. Stochastic approaches allow one to account for random perturbations and incomplete information regarding the state of the system [22, 23], and specialized methods for extreme value estimation in correlated observations may also be applied [23].

The methods developed in this paper draw on experience with mathematical tools for modeling the dynamics of the number of individuals with stress disorders and forecasting under uncertainty [7], for prioritizing soldier resilience competencies using fuzzy logic [15, 16], and for the analysis of conflicts using data-driven methods [8]. The contemporary context of the problem is also shaped by practical questions of defence spending among NATO member states and pressure to meet spending targets [9, 10], as well as the destabilizing influence of geopolitical tensions [14].

In the present paper, an approach is employed that combines methods of antagonistic game theory and convex set analysis. The unknown function describing the armament level of the second country is assumed to belong to given compact, centrally symmetric convex sets. This allows the problem of finding the optimal strategy of the first country to be formulated as a constrained optimization problem, for which analytical relations are derived. Additionally, the case of stochastic uncertainty is considered, where the system parameters are pairwise uncorrelated random variables, and the corresponding variance estimates are proved.

3. Main Results

Let $x_1(k)$ and $x_2(k)$ be the amount of money spent by countries on armaments, respectively. According to Richardson's model [5], the dynamics of funds over time, namely, at a discrete moment of time, can be described by a system of difference equations:

$$\begin{aligned} x_1(k+1) &= x_1(k) - \alpha x_1(k) + \beta x_2(k) + u_1(k), \\ x_2(k+1) &= x_2(k) - \gamma x_1(k) + \delta x_1(k) + u_2(k), \end{aligned} \qquad (1)$$

where $u_1(k), u_2(k)$ are investments in the defense sectors; $\alpha, \beta, \gamma, \delta$ are positive parameters.

Let us write system (1) in vector form. Let

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \quad A = \begin{pmatrix} 1-\alpha & \beta \\ \delta & 1-\gamma \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we obtain an equation

$$x(k+1) = Ax(k) + e_1 u_1(k) + e_2 u_2(k),$$
$$x(0) = x_0. \tag{2}$$

Let us denote a vector $\vec{u}_2 = \left((x(0), e_2), u_2(0), ..., u_2(N-1)\right)^T$. We assume that the vector $\vec{u}_2$ is unknown and belongs to a family of compact sets $G_\lambda, \lambda \in (0, \infty)$. We also assume further that G_λ are convex, centrally symmetric sets with respect to the given vector $\bar{u}_2 = \left((x(0), e_2), \bar{u}_2(0), ..., \bar{u}_2(N-1)\right)$. Suppose that a vector $\frac{1}{\lambda}\left(\vec{u}_2 - \bar{u}_2\right)$ belongs to a convex compact G_1 for arbitrary $\lambda > 0$.

Let us introduce a function $F(\vec{u}_1, \vec{u}_2)$ in the form

$F(\vec{u}_1, \vec{u}_2) = \left|(x(N), e_1) - C\right|,$

where $\vec{u}_1 = \left(u_1(0), u_1(1), ..., u_1(N-1)\right)^T$ is a vector with known components from system (1), C is a given constant.

Let us denote by $\lambda_{\max}(u_1)$ the maximum value of the parameter λ at which the inequality

$$\max_{\vec{u}_2 \in G_\lambda} \left|(x(N), e_1) - C\right| \le \mu,$$

holds, where μ is a given positive value.

Statement 1. The equality holds

$$\max_{\vec{u}_1 \in G_\lambda} \lambda_{\max}(\vec{u}_1) = \frac{\mu}{\max\limits_{G_1} (x^{(1)}(N), e_1)}, \tag{3}$$

where $x^{(1)}(k)$, $k = \overline{0, N-1}$, *are solutions of a system*

$$\begin{cases} x^{(1)}(k+1) = Ax^{(1)}(k) + e_2(u_2(k) - \bar{u}_2(k)), \\ x^{(1)}(0) = \left(x(0) - \bar{x}(0), e_2\right) e_2, \end{cases} \tag{4}$$

Proof. We represent the vector $x(k)$ in the form

$$x(k) = x^{(1)}(k) + x^{(2)}(k),$$

where $x^{(1)}(k)$ is the solution of system (4) and $x^{(2)}(k)$ is the solution of a system

$$\begin{cases} x^{(2)}(k+1) = Ax^{(2)}(k) + e_1 u_1(k) + e_2 \bar{u}_2(k)), \\ x^{(2)}(0) = (x(0), e_1) e_1 + (\bar{x}(0), e_2) e_2. \end{cases} \tag{5}$$

We further note that the equality holds

$$\max_{G_\lambda} \left|(x(N), e_1) - C\right| = \max_{G_1} \left|(x^{(1)}(N), e_1)\lambda + (x^{(2)}(N), e_1) - C\right| =$$
$$= \max_{G_1} \left(x^{(1)}(N), e_1\right)\lambda + \left|(x^{(2)}(N), e_1) - C\right|,$$

which means that such an inequality

$$\max_{G_\lambda} \left|(x(N), e_1) - C\right| \le \mu$$

holds if and only if the parameter λ satisfies the inequality

$$\lambda \le \frac{\mu - \left|(x^{(2)}(N), e_1) - C\right|}{\max\limits_{G_1} (x^{(1)}(N), e_1)}.$$

Therefore,

$$\lambda_{\max}(\vec{u}_1) = \frac{\mu - \left|(x^{(2)}(N), e_1) - C\right|}{\max\limits_{G_1} (x^{(1)}(N), e_1)}.$$

From here, we get that

$$\max_{\vec{u}_1} \lambda_{\max}(\vec{u}_1) = \frac{\mu - \min\limits_{\vec{u}_1} \left|(x^{(2)}(N), e_1) - C\right|}{\max\limits_{G_1} (x^{(1)}(N), e_1)}.$$

Now let us show that $\min_{\bar{u}_1}\left|(x^{(2)}(N),e_1)-C\right|=0$. Since the following equality holds

$$(x^{(2)}(N),e_1)=(\varphi(0),e_1)(x(0),e_1)+(\varphi(0),e_2)(\bar{x}(0),e_2)+$$
$$+\sum_{k=0}^{N-1}(e_1,\varphi(k+1))u_1(k)+\sum_{k=0}^{N-1}(e_2,\varphi(k+1))\bar{u}_2(k), \quad (6)$$

where $u_1(k),\ k=\overline{0,N-1}$, is given value from system (2), $\bar{u}_2(k),\ k=\overline{0,N-1}$ are components of given vector, $\varphi(k)$ satisfies a problem

$$\begin{cases}\varphi(k)=A^T\varphi(k+1),\\ \varphi(N)=e_1,\end{cases} \quad (7)$$

then we can choose the values $u_1(k),\ k=\overline{0,N-1}$ in equality (6) in the way

$$\sum_{k=0}^{N-1}(e_1,\varphi(k+1))u_1(k)=\tilde{C}, \quad (8)$$

where

$$\tilde{C}=-\Big[(\varphi(0),e_1)(x(0),e_1)+(\varphi(0),e_2)(\bar{x}(0),e_2)+$$
$$+\sum_{k=0}^{N-1}(e_2,\varphi(k+1))\bar{u}_2(k)\Big]+C.$$

The last considerations lead to equality (3). □

Remark 1. The set of all possible values $u_1(k),\ k=\overline{0,N-1}$ satisfying equality (8) has the form

$$u_1(k)=\hat{u}_1(k)+v(k), \quad (9)$$

where

$$\hat{u}_1(k)=\frac{\tilde{C}\psi(k)}{\sum_{k=0}^{N-1}\psi^2(k)}, \quad (10)$$

$\psi(k)=\left(e_1,\varphi(k+1)\right)$, and term $v(k)$ satisfies the equality

$$\sum_{k=0}^{N-1}\psi(k)v(k)=0.$$

In addition, due to the orthogonality of the terms of equality (9), the equality holds

$$\sum_{k=0}^{N-1}u_1^2(k)=\sum_{k=0}^{N-1}\hat{u}_1^2(k)+\sum_{k=0}^{N-1}v^2(k).$$

Consequence 1. If the set G_λ has the form

$$G_\lambda=\Big\{\bar{u}_2:\ \left|\left(x(0),e_2\right)-\left(\bar{x}(0),e_2\right)\right|\le\lambda C_0,$$
$$\left|u_2(0)-\bar{u}_2(0)\right|\le\lambda C_1,...\left|u_2(N-1)-\bar{u}_2(N-1)\right|\le\lambda C_N\Big\},$$

then

$$\max_{G_1}\left(x^{(1)}(N),e_1\right)=\max_{G_1}\Big((x(0),e_2)(\varphi(0),e_1)+$$
$$+\sum_{k=0}^{N-1}u_2(k)(\varphi(k+1),e_1)\Big)=$$
$$=\sum_{k=0}^{N-1}C_k\left|(\varphi(k+1),e_1)\right|+C_0\left|(\varphi(k+1),e_1)\right|,$$

where

$$G_1=\Big\{\left((x(0),e_2\right),u_2(0),...,u_2(N-1):\ \left|(x(0),e_2)\right|\le C_0,$$
$$\left|u_2(0)\right|\le C_1,...,\left|u_2(N-1)\right|\le C_N\Big\}.$$

Consequence 2. If the set G_λ has the form

$$G_\lambda = \Big\{ \vec{u}_2 : \ q_0^2 \left((x(0), e_2) - (\bar{x}(0), e_2) \right)^2 + $$
$$ + \sum_{k=0}^{N-1} q_{k+1}^2 \left(u_2(k) - \bar{u}_2(k) \right)^2 \le \lambda^2 \Big\},$$

where q_k, $k = \overline{0, N}$, are given constants, then the equality holds

$$\max_{G_1} \left(x^{(1)}(N), e_1 \right) =$$
$$= \left\{ q_0^{-2} (\varphi(0), e_1)^2 + \sum_{k=0}^{N-1} q_{k+1}^{-2}(k) (\varphi(k+1), e_1)^2 \right\}^{1/2} .$$

We assume that constraints are imposed on the vector $\vec{u}_1 = \left(u_1(0), u_1(1), ..., u_1(N-1) \right)^T$. Let

$$\vec{u}_1 \in U = \left\{ \vec{u}_1 : (\vec{u}_1, \vec{u}_1) = \sum_{k=0}^{N-1} u_1^2(k) \le \beta^2 \right\},$$

here β is given constant.

Statement 2. The equality holds

$$\max_{\vec{u}_1 \in U} \lambda_{\max}(\vec{u}_1) = \begin{cases} \dfrac{\mu}{\max\limits_{G_1}(x^{(1)}(N), e_1)}, \ if\, \beta^2 \ge (\hat{u}_1, \hat{u}_1), \\ \lambda_{\max}(\hat{\hat{u}}_1), \ if \ \beta^2 < (\hat{u}_1, \hat{u}_1), \end{cases} \tag{11}$$

where the values $\hat{u}_1(k)$ are chosen from equality (10), $\psi(k) = \left(e_1, \varphi(k+1) \right)$, vector $\varphi(k)$ is the solution of system (7), and

$$\hat{\hat{u}}_1(k) = -\tilde{C}\psi(k) \left| \frac{\tilde{C}}{\beta^2} \sum_{k=0}^{N-1} \psi^2(k) \right|^{-1/2}, \tag{12}$$

$$\lambda_{\max}(\hat{\hat{u}}_1) = \frac{|\tilde{C}| \left| \dfrac{\beta}{\tilde{C}} \left(\sum\limits_{k=0}^{N-1} \psi^2(k) \right)^{1/2} - 1 \right|}{\max\limits_{G_1}(x^{(1)}(N), e_1)} . \tag{13}$$

Proof. The first case in equality (11) when $\beta^2 \ge (\hat{u}_1, \hat{u}_1)$ follows from Statement 1.

Let us consider the case when $\beta^2 < (\hat{u}_1, \hat{u}_1)$ **.** Denote a function

$$f(\vec{u}_1) = \left| (x^{(2)}(N), e_1) - C \right| .$$

Solving the problem of the conditional extremum of function $f^2(\vec{u}_1)$ under the condition $\beta^2 \ge (\hat{u}_1, \hat{u}_1)$, we find a vector $\hat{u}_1$ with components $\hat{\hat{u}}_1(k)$ from (12).

Substituting $\hat{\hat{u}}_1$ into expression for $\lambda_{\max}(\vec{u}_1)$, we obtain the value $\lambda_{\max}(\hat{\hat{u}}_1)$ in the form (13). □

Let us proceed to consider the case when the controls $u_1(k)$ in system (1), which according to Richardson's model mean the investment of the first country in armaments, has the feedback form. Next, we assume that the control $u_1(k)$ has the form

$$u_1(k) = v_1(k)x_1(k) + v_2(k)x_2(k) + v_0(k), \tag{14}$$

where $v_i(k)$, $i = \overline{0, 2}$, are sequences that we choose from the condition

$$\left(v_0(k), v_1(k), v_2(k) \right) \in Arg \ \max \lambda_{\max}(\vec{u}_1) .$$

Here and further we mean

$$\lambda_{\max}(\vec{u}_1) = \lambda_{\max}(v_1, v_2, v_0) \,. \tag{15}$$

We suppose that there exist the sets $G^{(1)}$ and $G^{(2)}$ such that $G^{(1)} \subseteq G_1 \subseteq G^{(2)}$ and they have the form

$$G^{(i)} = \left\{ \vec{u}_2 : \left(Q^{(i)} \vec{u}_2, \vec{u}_2 \right) \le 1 \right\}, \; i = 1, 2 \,, \tag{16}$$

where $Q^{(i)}$ are positive definite matrices of the form

$$Q^{(i)} = \begin{pmatrix} q_{0,i}^2 & 0 & \vdots & 0 \\ 0 & q_{1,i}^2 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & q_{N,i}^2 \end{pmatrix}.$$

Let us denote by A_1 the matrix of the form

$$A_1 = A + \begin{pmatrix} v_1(k) & v_2(k) \\ 0 & 0 \end{pmatrix},$$

where the matrix A is given in equation (2), the values $v_i(k), \; i = \overline{0,2}$ are from (14).

For the matrix A_1 defined in this way and for the control $u_1(k)$ given by formula (14), for the vector $x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$ the original problem can be written as a vector equation of the following form

$$\begin{gathered} x(k+1) = A_1 x(k) + e_1 v_0(k) + e_2 u_2(k), \\ x(0) = x_0 . \end{gathered} \tag{17}$$

Let us introduce a sequence of vector random variables $\zeta^{(i)}(k) = \begin{pmatrix} \zeta_1^{(i)}(k) \\ \zeta_2^{(i)}(k) \end{pmatrix}, \; k = \overline{0, N-1}, \;\; i = 1, 2 \,,$ as solutions to the problem

$$\begin{gathered} \zeta^{(i)}(k+1) = A_1 \zeta^{(i)}(k) + e_2 \xi_2^{(i)}(k), \\ \zeta^{(i)}(0) = e_2 \eta_{0,i} , \end{gathered} \tag{18}$$

here $\eta_{0,i}, \xi_2^{(i)}(0), ..., \xi_2^{(i)}(N-1)$ are some random variables.

Statement 3. *Let in problem (18) $\eta_{0,i}, \xi_2^{(i)}(0), ..., \xi_2^{(i)}(N-1)$ be a sequence of pairwise uncorrelated random variables with $i = 1, 2$ and such that the conditions are fulfilled*

$$\begin{gathered} E\eta_{0,i} = 0 \,, \; E\eta_{0,i}^2 = q_{0,i}^{-2} \,, \\ E\xi_2^{(i)}(k) = 0 \,, \; E(\xi_2^{(i)}(k))^2 = q_{k+1,i}^2 \,, \;\; k = \overline{0, N-1} . \end{gathered}$$

Then the following inequalities hold

$$\bar{\mu} \cdot \left(P_2(N) e_2, e_2 \right)^{-1/2} \le \lambda_{\max}(\vec{u}_1) \le \bar{\mu} \cdot \left(P_1(N) e_2, e_2 \right)^{-1/2} , \tag{19}$$

where $\bar{\mu} = \mu - \left| (x^{(2)}(N), e_1) - C \right|$, *matrices* $P_i(k), i = 1, 2$ *are solutions of matrix difference equations*

$$\begin{gathered} P_i(k+1) = A_1 P_i(k) A_1^T + e_2 e_2^T q_{k+1,i}^{-2} , \\ P_i(0) = q_{0,i}^{-2} e_2 e_2^T . \end{gathered} \tag{20}$$

Proof. Since $\lambda_{\max}(\vec{u}_1) = \bar{\mu} \left(\max\limits_{G_1} (x^{(1)}(N), e_2) \right)^{-1}$, then from the inequality

$$\max_{G^{(1)}}(x^{(1)}(N),e_2) \le \max_{G_1}(x^{(1)}(N),e_2) \le \max_{G^{(2)}}(x^{(1)}(N),e_2)$$

yields the following inequality

$$\bar{\mu}\left(\max_{G^{(2)}}(x^{(1)}(N),e_2)\right)^{-1} \le \lambda_{\max}(\vec{u}_1) \le \bar{\mu}\left(\max_{G^{(1)}}(x^{(1)}(N),e_2)\right)^{-1}.$$

We further note that since $x^{(1)}(k)$ is a solution of the equation

$$x^{(1)}(k+1) = A_1 x^{(1)}(k) + u_2(k)e_2, \quad x^{(1)}(0) = (x_0,e_2)e_2,$$

then

$$(x^{(1)}(N),e_2) = (\varphi_1(0),e_2)(x_0,e_2) + \sum_{k=0}^{N-1}(\varphi_1(k+1),e_2)u_2(k),$$

where $\varphi_1(k)$ is the solution of problem $\varphi_1(k) = A_1^T\varphi_1(k+1),\ \varphi_1(N) = e_2$. From the generalized Cauchy-Bunyakovsky inequality it follows that

$$\max_{G^{(i)}}(x^{(1)}(N),e_2) =$$
$$= \left\{(\varphi_1(0),e_2)q_{0,i}^{-2} + \sum_{k=0}^{N-1}(\varphi_1(k+1),e_2)q_{k+1,i}^{-2}\right\}^{1/2} =$$
$$= \left\{E(\varsigma^{(i)}(N),e_2)^2\right\}^{1/2}.$$

Since $E(\varsigma^{(i)}(N),e_2)^2 = (P_i(N)e_2,e_2),\ i=1,2$, де $P_i(N)$ is the solution of problem (20), then inequalities (19) hold. □

Let us next introduce the function $F(V_1,V_2)$

$$F(V_1,V_2) = (P_2(N)e_2,e_2) + \nu^2\left[\sum_{k=0}^{N-1}\Big(V_1^2(k)(P_2(k)e_1,e_1) + \right.$$
$$\left. + V_2^2(P_2(k)e_2,e_2) + 2V_1(k)V_2(k)(P_2(k)e_2,e_2)(P_2(k)e_2,e_2)\Big)\right].$$

Lemma. *The equality holds* $\min_{V_1,V_2} F(V_1,V_2) = F(\hat{V}_1,\hat{V}_2)$, *where*

$$\hat{V}(k) = \begin{pmatrix} V_1(k) \\ V_2(k) \end{pmatrix} = (\nu^2 + (S_{k+1}e_1,e_1))^{-1}e_1^T S_{k+1}A, \tag{21}$$

and matrix S_k *is a solution of the problem*

$$S_k = A^T S_{k+1}A - (\nu^2 + (S_{k+1}e_1,e_1))^{-1}A^T S_{k+1}e_1e_1^T S_{k+1}A, \tag{22}$$
$$S_N = e_1e_1^T.$$

Proof. Note that the equality holds

$$F(V_1,V_2) = E(\varsigma^{(2)}(N),e_2)^2 +$$
$$+\nu^2\sum_{k=0}^{N-1}E\Big(V_1(k)\varsigma_1^{(2)}(k) + V_2(k)\varsigma_2^{(2)}\Big)^2,$$

where $\varsigma^{(2)}(k) = \begin{pmatrix} \varsigma_1^{(2)}(k) \\ \varsigma_2^{(2)}(k) \end{pmatrix}$. We denote the vector $\eta(k) = \begin{pmatrix} \eta_1(k) \\ \eta_2(k) \end{pmatrix}$ as a solution to the equation

$$\eta(k+1) = A\eta(k) + u_1(k)e_1 + \xi(k)e_2,\ \eta(0) = \eta_{0,2}e_2$$

with the objective functional $J(\vec{u}_1) = E(\eta(N), e_2)^2 + \nu^2 \sum_{k=0}^{N-1} E u_1^2(k)$. The feedback control problem has a solution

$$\hat{u}_1(k) = -(\hat{V}(k), \eta(k))$$

and $J(\hat{u}) = \min_{V_1, V_2} F(V_1, V_2) = F(\hat{V}_1, \hat{V}_2)$, *here* $\hat{V}(k)$ has the form (21).

Remark. *The equality holds*

$$F(\hat{V}_1, \hat{V}_2) = (S_0 e_2, e_2) q_{0,2}^{-2} + \sum_{k=0}^{N-1} (S_{k+1} e_2, e_2) q_{k+1,2}^{-2} .$$

Statement 4. *Let* $\hat{V}_1(k), \hat{V}_2(k), k = \overline{0, N-1}$ *be sequences of form (21). Then the inequalities hold*

$$\frac{\mu - \min_{V_0} \left| (\hat{x}^{(2)}(N), e_1) - c \right|}{F(\hat{V}_1, \hat{V}_2)} \le \max_{V_0, V_1, V_2} \lambda_{\max}(V_0, V_1, V_2) \le$$

$$\le \frac{\mu}{\left\{ \min_{V_1, V_2} (P_2(N) e_2, e_2) \right\}^{1/2}},$$

where $\hat{x}_2(k+1) = A_1 \hat{x}_2(k) + V_0(k) e_1 + u_0(k) e_2$, $\hat{x}(0) = x_0 e_1 + \overline{x}_{0,2} e_2$.
The proof of Statement 4 follows from the above statements.

4. Numerical Experiment

Furthermore, to illustrate the above approach, we will conduct a numerical experiment using data from the open Military Expenditure Database from the Stockholm International Peace Research Institute (SIPRI) (**https://www.sipri.org/databases/milex**). The SIPRI (Stockholm International Peace Research Institute) website publishes international statistical data on military spending for all countries worldwide.

Since the financing of armaments for both countries increased significantly in 2022, the forecast estimates based on our approach have a large margin of error. Therefore, we take information to test our approach starting from 2022.

From the SIPRI Military Expenditure Database, we will use data on the defense budgets of Ukraine and Russia, expressed in billions of US dollars (USD) (Table 1).

Table 1: The defense budgets of Ukraine and Russia

Year	Ukraine, billion USD	Russia, billion USD
2022	41,18	88
2023	64,7	105
2024	75	145
2025	84	175
2026	95,5 (planned)	217 (planned)

Here, from 2022 to 2025, the actual amounts of financing for armaments are given, and in 2026, the financing planned in the budgets for both countries.

Let's put the following parameters in system (1):

$$1-\alpha = 0,92, \ \ \beta = 0,1; \ \ 1-\gamma = 0,99; \ \ \delta = 0,6; \tag{23}$$

$$u_1 = 0, \quad u_2(k) = -5 .$$

Based on the data for 2022-2026, we will construct a financing forecast for armaments for both countries for 2027- 2028.

Let us put $k = 0$ for 2026, and $u_1(0) = u_0$, $u_2(0) = -5$, then we can write under the data in Table 1 that

$$x_1(0) = 95,5, \quad x_2(0) = 217 \quad \text{(billion USD)}.$$

Then, following the algorithm above, we calculate a financing forecast for armaments for both countries for 2027 (here $k = 1$):

$$\begin{aligned} x_1(1) &= (1-\alpha)x_1(0) + \beta x_2(0) + u_1(0), \\ x_2(1) &= (1-\gamma)x_2(0) + \delta x_1(0) + u_2(0). \end{aligned} \tag{24}$$

Substituting the parameters given in (23) into these formulas, we obtain

$$\begin{aligned} x_1(1) &= 0,92 \cdot 95,5 + 0,1 \cdot 217 + u_0 = 109,56 + u_0, \\ x_2(1) &= 0,99 \cdot 217 + 0,6 \cdot 95,5 - 5 = 267,13. \end{aligned}$$

Next, setting $k = 2$, $u_1(1) = u_1$, and using formulas (24), we calculate the forecast for Ukraine and Russia for 2028

$$\begin{aligned} x_1(2) &= 0,92 \cdot (109,56 + u_0) + 0,1 \cdot 267,13 + u_1 = 0,92u_0 + u_1 + 127,51, \\ x_2(2) &= 0,99 \cdot 267,13 + 0,6 \cdot (109,56 + u_0) - 5 = 0,6u_0 + 325,2. \end{aligned} \tag{25}$$

With these parameters, the graph of changes in actual financing for armaments of both countries from 2022 to 2026 and forecast for 2027-2028 takes the following form (Pic. 1)

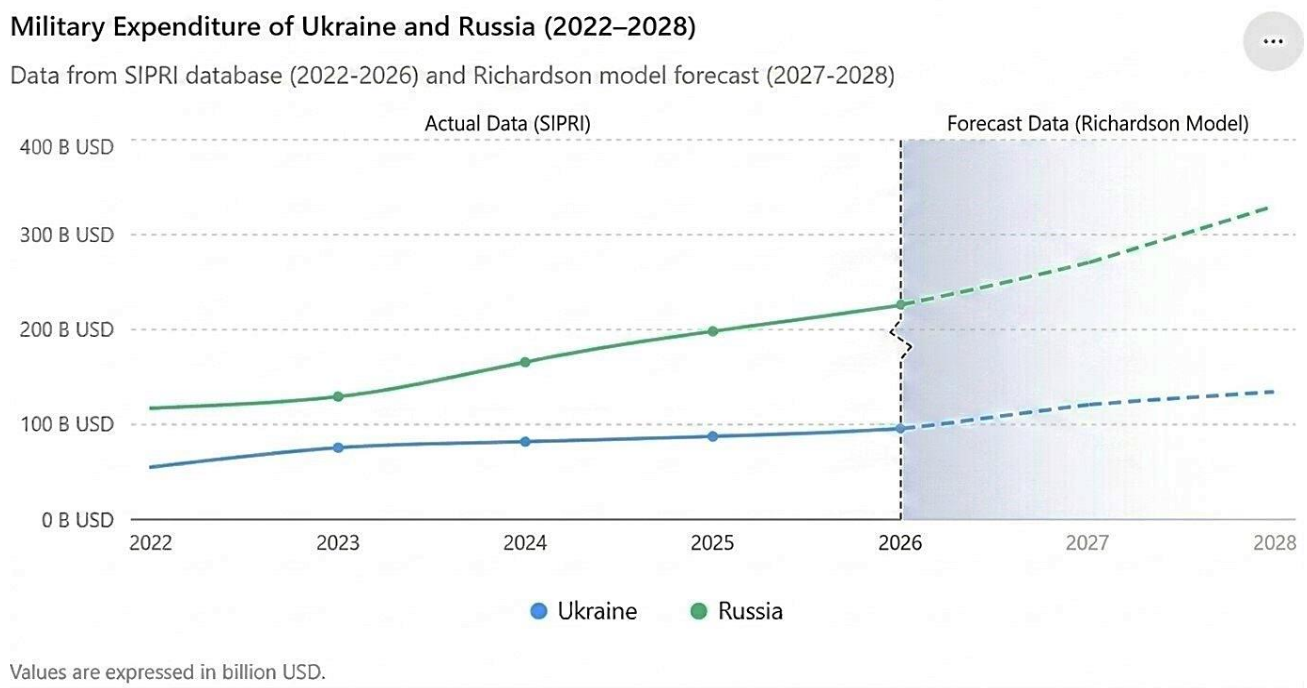

Fig. 1. Actual financing and forecast for armaments of both countries

Let us denote by $\tilde{x}_1$, $\tilde{x}_2$ the actual armaments financing for both countries respectively, in 2026. Then the deviation from the planned financing for Russia (217 billion USD) has the form

$$|\tilde{x}_2(0) - 217| \le \lambda .$$

In summary, in order for Ukraine's armaments spending to deviate from the average expected by no more than μ , provided that Russia's armaments spending in 2026 increases by no more than λ , it is necessary that Russia's spending does not exceed $\lambda_{\max}$.

We believe that according to military experts estimates, financing for Ukraine's armaments in 2028 will be within $125 \le x_1(2) \le 135$. In order to ensure that Ukraine has a maximum deviation λ of actual funding from the planned level of no more than 1 billion USD, the following condition must be met:

$$\max_{G_\lambda}\left|\tilde{x}_1(2)-C\right| \le \mu = 1,$$

where $\tilde{x}_1(2)$ can be calculated using (25)

$$\tilde{x}_1(2) = x_1(2) + ((1-\alpha)+(1-\gamma))\beta\left(x_2(0)-217\right),$$

the set G_λ has the form

$$G_\lambda = \left\{x_2(0):\ \left|x_2(0)-217\right| \le \lambda\right\},$$

C is positive value calculated as

$$C = \frac{125+135}{2} = 130.$$

Under our approach, using (24), (25), since

$$\max_{G_\lambda}\left|\tilde{x}_1(2)-C\right| = ((1-\alpha)+(1-\gamma))\beta\cdot\lambda + \left|x_1(2)-130\right| =$$

$$= 0{,}19\lambda + \left|0{,}92u_0 + u_1 - 2{,}49\right|,$$

then, denoting $v = 0{,}92u_0 + u_1$, we obtain the maximum deviation $\lambda_{\max}$ of the value λ from the formula

$$\lambda_{\max} = \frac{\mu - \left|\tilde{x}_1(2)-C\right|}{(1-\alpha)\beta + \beta(1-\gamma)} = \frac{1-\left|v-2{,}49\right|}{0{,}19}. \qquad (26)$$

It is obvious from (26) that the deviation $\lambda_{\max}$ reaches its maximum value under the condition $v = 2{,}49$. Then

$$\lambda_{\max} = \frac{1}{0{,}19} \approx 5{,}26.$$

Then the equality $\max_{G_\lambda}\left|\tilde{x}_1(2)-C\right| = 1$ holds and our algorithm establishes that $129 \le \tilde{x}_1(2) \le 131$.

In other cases, when $\left|v-2{,}49\right| \ne 0$, the maximum deviation $\lambda_{\max}$ takes on smaller values. Thus, if the deviation in financing the Russian defense sector $\left|\tilde{x}_2(0)-217\right| \le 5{,}26$, it means that actual armaments financing for Russia lies within the interval $211{,}74 \le x_2(0) \le 222{,}26$. In this case, Ukraine's financing x_1 will lie within the interval of expert estimates.

Conclusions

To study the dynamics described by Richardson models under uncertainty, it is also possible to use the methods of the theory of antagonistic games. In this theory, countries act as two players with their own strategies. In conditions when agreements between countries are not observed, and therefore, Nash strategies are ineffective, cautious strategies are used. The article showed that, by applying cautious strategies under conditions of uncertainty, it is possible to calculate the maximum level of armaments of the second country, at which the first country will provide itself with a level of armaments that deviates by a given value. When solving this problem, strategies of the first country that depend only on time, as well as strategies that depend on the size of armaments at a given time, are used.

Acknowledgements. This work has been partially supported by the Ministry of Education and Science of Ukraine: R&D Project "Development of new mathematical methods for optimization problems under uncertainty" (registration number 0125U002257) for the period of 2025-2027 at the Taras Shevchenko National University of Kyiv.

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